J. Energy Conversion

نشریه علمی-تخصصی تبدیل انرژی (JEED)

Volume: 12, Issue:3, 2025: 53-72

دوره ۱۲، شماره ۳، پاییز ۱۴۰۴، ص: ۷۲-۵۳

Development and Implementation of a High-Order Compact Finite Difference Method in Numerical Analysis of Fluid Flows

Mohamad Hamed Hekmat

Department of Mechanical Engineering, Tafresh University, Tafresh, Iran

Extended Abstract

Introduction

Numerical simulation of fluid-flow problems strongly depends on the quality of the computational grid used to discretize the physical domain. In structured-grid approaches, finite-difference methods provide a relatively straightforward framework for approximating the governing partial differential equations; however, conventional finite-difference formulations generally require a trade-off between numerical accuracy and computational cost. Increasing the order of accuracy commonly requires the use of a larger computational stencil, which increases the bandwidth of the resulting algebraic system and consequently raises computational requirements. High-order compact finite difference (HOCFD) schemes have been developed to overcome this limitation by achieving higher-order accuracy while maintaining a relatively compact computational stencil. In the present study, a fourth-order high-order compact finite difference formulation is developed and implemented for elliptic grid generation. The main objective is to improve the efficiency of structured-grid generation while preserving high grid quality, particularly in problems involving local grid clustering and orthogonality near boundaries. Elliptic grid-generation methods are advantageous because they provide smooth distributions of grid points and allow effective control of clustering and boundary orthogonality, although their major drawback is the relatively high computational cost associated with solving the governing elliptic equations. The proposed approach is therefore intended to reduce the computational cost of grid generation without sacrificing the quality of the resulting grid. Its performance is evaluated for both internal and external flow configurations and is compared with the conventional second-order central finite difference method. The ultimate objective is to establish an efficient numerical procedure that can be employed in computational fluid dynamics simulations involving complex and fine computational grids.

2. Materials and Methods

The grid-generation procedure is based on elliptic partial differential equations formulated in a computational domain and transformed into the physical domain. Laplace- and Poisson-type equations are employed to determine the physical coordinates of the grid points. The Poisson formulation is particularly important because it permits control over the spatial distribution of grid points. Source functions are introduced to generate clustering of grid points near prescribed lines or points and to control the orthogonality of grid lines relative to arbitrary boundaries. An algebraic grid is initially generated and used as the starting distribution for the elliptic grid-generation procedure. The governing equations are then discretized and solved iteratively using the Gauss–Seidel method. Grid clustering is controlled

[Downloaded from jeed.dezful.iau.ir on 2026-09-22]

through source functions containing amplification and damping parameters, allowing the computational points to be concentrated around selected lines or locations. In addition, a specifically formulated orthogonality function is incorporated to improve the alignment of grid lines with the physical boundaries. This is particularly important for boundary-layer-type problems, where accurate evaluation of gradients normal to the wall is strongly dependent on grid orthogonality. The principal numerical development concerns the discretization of the elliptic grid-generation equations using a fourth-order compact finite difference formulation. In this approach, truncation-error terms generated by the conventional central finite difference approximation are expressed in terms of the governing differential equation itself. Through successive differentiation and substitution of the governing equation, these error terms are incorporated into a compact formulation, allowing fourth-order accuracy to be achieved without introducing a large computational stencil. A major advantage of the proposed formulation is demonstrated by the computational stencil. The fourth-order compact scheme requires 13 grid points, whereas a conventional fourth-order central finite difference approximation requires 25 points. Furthermore, when the right-hand side of the compact formulation is evaluated explicitly from the previous iteration, the number of required points can be reduced to 9, yielding a fully compact formulation. The resulting discrete equations are solved using the Gauss–Seidel iterative method. The developed method was assessed through several test cases. Internal-flow grid generation was examined using a square computational domain and prescribed clustering near boundaries and corner points. External-flow grid generation was subsequently investigated around a NACA0012 airfoil at zero angle of attack. Finally, the accuracy of the proposed grid-generation approach was assessed through an aerodynamic simulation of flow over the NACA0012 airfoil at a free-stream Mach number of 0.8 and an angle of attack of approximately 1.49°, using a 160 × 80 computational grid.

3. Results and Discussion

The qualitative assessment of the generated grids demonstrated that the proposed fourth-order compact formulation can produce smooth and well-organized computational grids around complex geometries. For the NACA0012 airfoil, a 20 × 41 grid was generated using the elliptic formulation and the proposed compact scheme. Approximately 92% of the grid intersection angles were located within the range of 85°–95°, indicating a high degree of grid orthogonality. In addition, the variation in the lengths of neighboring cells was reported to be less than 1.1, demonstrating a smooth and relatively uniform distribution of grid points. These characteristics are important for reducing numerical errors, particularly in regions close to solid boundaries. Quantitative comparisons between the fourth-order compact method and the conventional central finite difference method demonstrated a substantial reduction in computational effort. For a 20 × 20 internal-flow grid, the standard central scheme required 364 iterations, whereas the compact scheme required only 36 iterations, corresponding to approximately 90% reduction in computational time. For 30 × 30 and 40 × 40 grids, the numbers of iterations for the standard method were 201 and 94, respectively, compared with 23 and 9 iterations for the compact formulation. The corresponding computational-time reductions were approximately 88% and 80%, respectively. The convergence behavior further confirmed the computational advantage of the proposed formulation. The residual decreased considerably faster when the fourth-order compact method was used. In the examined internal-flow cases, the error of the compact formulation decreased sharply during the initial iterations and approached zero at approximately the ninth iteration, whereas the conventional central finite difference method exhibited substantially slower convergence. This behavior demonstrates that the increased numerical accuracy of the compact fourth-order formulation directly contributes to faster convergence of the grid-generation procedure. The method was also evaluated for external-flow grid generation around a NACA0012 airfoil. For the first external-flow configuration, the compact method reduced the number of iterations from 861 to 230, corresponding to approximately 74% reduction in computational time. For the second configuration, the required iterations decreased from 602 to 222, yielding a reduction of approximately 63%. The convergence plots showed that the compact method reached the specified residual level substantially earlier than the conventional scheme. For one of the test cases, the compact method reached a residual below 10^{-3} at approximately the 106th iteration, whereas the conventional method required approximately 194 iterations. The benefits of the compact method became particularly evident when the external-flow grids were refined. For the third grid configuration, increasing the grid resolution from 25 × 25 to 35 × 35 and 45 × 45 resulted in computational-time reductions of approximately 81%, 76%, and 71%, respectively, compared with the standard central finite difference method. For the fourth configuration, the corresponding reductions were approximately 80%, 75%, and 64%. These results indicate that the compact formulation remains highly effective as the computational grid becomes finer, which is particularly valuable in simulations requiring detailed resolution of boundary layers and other regions with strong gradients. To verify that the improved grid-generation procedure did not adversely affect the physical accuracy of the subsequent fluid-flow solution, an aerodynamic simulation of flow over a NACA0012 airfoil was performed. The flow conditions corresponded to a free-stream Mach number of 0.8 and

[Downloaded from jeed.dezful.iau.ir on 2026-09-22]

an angle of attack of approximately 1.49°, using a 160 × 80 computational grid. The pressure coefficient distribution obtained using the compact grid showed very good agreement with both the conventional-grid solution and available experimental data. The comparison therefore demonstrated that the computational efficiency gained during grid generation did not compromise the accuracy of the final aerodynamic predictions. The aerodynamic coefficients obtained using the two grid-generation approaches were also highly consistent. The standard central finite difference method predicted a lift coefficient of 0.0033 and a drag coefficient of 0.2621, whereas the proposed compact formulation yielded values of 0.0035 and 0.2630, respectively. The close agreement between these values confirms that the proposed method produces a computational grid capable of providing reliable aerodynamic predictions while substantially reducing the computational effort required for grid generation. Overall, the numerical results indicate that the principal advantage of the proposed approach is not a change in the final physical solution but a considerable improvement in the efficiency of the grid-generation stage. By increasing the discretization accuracy from second to fourth order while maintaining a compact stencil, the method simultaneously reduces the number of iterations required for convergence and the computational operations associated with each iteration. This makes the approach particularly promising for fine, complex grids and applications in which grid generation must be repeated many times, such as iterative design and numerical optimization.

4. Conclusions

In this study, a fourth-order high-order compact finite difference method was developed and implemented for elliptic structured-grid generation in numerical fluid-flow analysis. The governing elliptic equations were discretized using a compact fourth-order formulation and solved iteratively using the Gauss–Seidel method. The proposed formulation was designed to provide higher numerical accuracy while requiring substantially fewer computational points than conventional high-order central finite difference schemes. In particular, the fourth-order compact formulation required 13 points compared with 25 points for the corresponding conventional fourth-order central formulation, with the possibility of reducing the stencil to 9 points under a lagged right-hand-side treatment. The results demonstrated that the proposed method significantly improves the efficiency of elliptic grid generation. For internal-flow cases, computational-time reductions of up to approximately 90% were obtained, while external-flow cases exhibited reductions of up to approximately 81%. The method also produced grids with high orthogonality and smooth point distributions; for the NACA0012 test case, approximately 92% of the grid angles were within 85°–95°, confirming the high quality of the generated grid. The aerodynamic validation further demonstrated that the proposed approach preserves the accuracy of the final fluid-flow solution. The pressure coefficient distribution obtained with the compact grid agreed well with both the conventional numerical solution and experimental data. Moreover, the lift and drag coefficients obtained using the compact formulation (0.0035 and 0.2630) were very close to those obtained using the standard method (0.0033 and 0.2621). Therefore, the developed fourth-order compact finite difference method can be considered an efficient and accurate approach for structured-grid generation in computational fluid dynamics. Its ability to reduce computational effort while maintaining grid quality and the accuracy of subsequent flow predictions makes it particularly suitable for complex three-dimensional simulations, fine-grid applications, and iterative optimization problems in which computational grids must be generated repeatedly. Future extensions of the method toward more complex three-dimensional configurations, highly refined grids, and multi-stage optimization procedures could further demonstrate its computational advantages.

Key words: elliptic grid generation, higher order compact finite difference method, standard finite difference method, computational cost

corresponding author: hekmat@tafreshu.ac.ir

Cite this article as: Mohamad Hamed Hekmat, Optimization Development and Implementation of a High-Order Compact Finite Difference Method in Numerical Analysis of Fluid Flows, **Journal of Energy Conversion, 2025, 12(3), 53-72**.

[Downloaded from jeed.dezful.iau.ir on 2026-09-22]